OJ -ALGEBRA OF MEASURABLE ELEMENTS WITH RESPECT TO A SUBADDITIVE MEASURE ON JORDAN ALGEBRAS

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Annotation

This article proves that the topological Jordan algebra of measurable elements with respect to sub additive measure is an OJ - algebra.

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Introduction

Let **A** be a finite *JBW* algebra, τ be an exact normal finite trace on **A**. Let *m* be a subadditive measure on **A** . From the results [2-3] it follows that *m* can be represented as $m(x)=\gamma(\tau(x))$. Let *N* be the space of normal functionals on **A**.

Lemma 1. Set $$S=\bigcup_{n=1}^{\infty}\{g\in N:-nm\le g\le nm \ \ на \ \ \nabla\}$$

is dense in the Banach space *N* , where $g\le nm$ on ∇ means that $g(e)\le nm(e)$ for any $e\in\nabla$.

Proof . If *S* is not dense in *N* , then there exists a continuous linear functional x_0 on *N* such that $x_0\neq 0$. $g(x_0)=0$ for everyone $g\in S$. Since it is $g(x_0)=0$ equivalent to the equality $g(r(x_0))=0$, where $r(x_0)$ is the support of the element x_0, it suffices to prove that $r(x_0)=0$. It is easy to see that $\tau\le m$ on ∇. The functional $\tau_e(x)=\tau(ex)$ also belongs to the set *S* . By assumption $g(r(x_0))=0$, for any $g\in S$ and in particular $\tau_e(r(x_0))=0, \ \forall e\in\nabla$. Letting $e=r(x_0)$ we have that $\tau(r(x_0))=0$. Due to accuracy, we τ conclude that $r(x_0)=0$. This means that $x_0=0$. Therefore, $x_0=0$. The lemma is proven.

Let **A** - *JBW* - algebra, ∇ be the set of idempotents of **A**. *m* is a finite subadditive measure on **A** , *t* is the topology of convergence in measure *m* .

Theorem 1. If the sequence of elements $\{x_n\}\subset\mathbf{A}$ *t* – c tends to zero and is bounded on the norm $(\|x_n\|\le 1, \ n=1,2,....)$, then it $*$- weakly converges to zero in **A** . **Proof.** Let $x_n\xrightarrow{t}x$ i.e. for any $\varepsilon,\delta>0$ there is a number n_0 such that $x_n\in N(\varepsilon,\delta)$ for $n\ge n_0$. This means that there is a sequence $\{e_n\}\subset\nabla$ such that $m(e_n^{\perp})\le\delta, \ \|U_e x_n\|\le\varepsilon, \ n\ge n_0$.

It needs to be shown that $x_0\to\theta$ $*$- weakly, i.e. $g(x_n)\to 0$ for any normal state $g\in N$. Let first $g\in S$, i.e. $g(e_n^{\perp})\le k_0 m(e_n^{\perp})$ for some natural k_0. We have:

$$g(x_n)=g\left(U_{e_n}x_n\right)+2g\left(U_{e_n,1-e_n}x_n\right)+(U_{1-e_n}x_n)$$

and $|g(U_{e_n}x_n)|\leq\|U_{e_n}x_n\|g(1)\leq\varepsilon$. Let us estimate the second term. Because $U_{e_n,1-e_n}x_n=2(1-e_n)(e_nx_n)$, then due to the Schwartz inequality

$$|g(U_{e_n,1-e_n}x_n)|\leq 2\sqrt{g(e^{\perp})g((e_nx_n)^2)}\leq$$

$$\leq 2\sqrt{g(e^{\perp})}\sqrt{\|e_nx_n\|^2}\leq 2\sqrt{k_0m(e^{\perp})}\|x_n\|\leq 2\sqrt{k_0}\sqrt{\delta}.$$

Taking into account the equality $U_{1-e_n}x_n=(1-e_n)(x_n-2e_nx_n)$, it similarly turns out that $|g(U_{1-e_n}x_n)|\leq 3\sqrt{k_0}\sqrt{\delta}$. By virtue of arbitrariness, ε,δ this implies that $g(x_n)\to 0$.

Let now $f\in N$ be an arbitrary normal state. By Lemma 1, for any $\eta>0$ there exists $g\in S$ such that $\|f-g\|\leq\eta$. Then if $g(e)\leq k_0m(e)$, then for $n\geq n_0$ we have:

$|f(x_n)|\leq|(f-g)(x_n)|+|f(x_n)|\leq\|f-g\|\|x_n\|++|g(x_n)|\leq\eta+\varepsilon+7\sqrt{k_0}\sqrt{\delta}$, i.e. $|f(x_n)|\to 0$. So, it $x_n\to 0$ $*$- weakly. The theorem has been proven.

Theorem 2 . The algebra $\widehat{\mathbf{A}}$ is a universal *OJ* - algebra, the set of bounded elements of which coincides with **A** **Proof**. In terms of continuity in the topology *t* of the operation of multiplication in $\widehat{\mathbf{A}}$, the set of $\widehat{\mathbf{A}}^+$ all squares of elements from $\widehat{\mathbf{A}}$ is the *t* - closure of the cone $\mathbf{A}^+=\{a^2,\ a\in\mathbf{A}\}$ *JBW* are algebras **A** . The cone $\widehat{\mathbf{A}}^+$ defines a $\widehat{\mathbf{A}}$ partial order, which obviously satisfies axioms 1), 2), 4) of the definition of a partial order and induces the initial partial order on **A.**

The proof of the second part of the theorem (i.e., the set of bounded elements of $\widehat{\mathbf{A}}$ which coincides with **A**) is carried out similarly to the proof of the theorem from [1].

Let be $\widehat{\mathbf{A}}_0$ an arbitrary maximal strongly associative subalgebra $\widehat{\mathbf{A}}$. Due to *t being the* continuity of multiplication in $\widehat{\mathbf{A}}$, subalgebra $\widehat{\mathbf{A}}_0$ closed. Let $K=\{a\in\widehat{\mathbf{A}}_0, a\geq 0\}$. The set of elements of the form $(1+x)^{-1},\ x\in K$, is contained, as noted earlier, in **A** . Since all are $x\in\widehat{\mathbf{A}}_0$ compatible, then by Lemma 1.3.2 from [1] the family is $\{(1+x)^{-1}, x\in K\}$ compatible. Let **A₀** be a maximal strongly associative subalgebra **A** containing this family. By virtue of the corollary of Theorem 1.2.2 . in [1], **A₀** is a topological semifield. If $\bar{A}_0$ the closure **A₀** in $\widehat{\mathbf{A}}$, then due to completeness $\widehat{\mathbf{A}}$, $\widehat{\mathbf{A}}_o$ is a complete topological semifield and hence a universal semifield . Obviously, it is $\bar{A}_0$ strongly associative in $\widehat{\mathbf{A}}$. Let's show that $\bar{A}_0=\widehat{\mathbf{A}}_0$. Since $\widehat{\mathbf{A}}_o$, it suffices to check that $\widehat{\mathbf{A}}_0\subset\bar{A}_0$.

Let $x\in K$, then $(1+x)^{-1}\in\mathbf{A}_0$ by definition **A₀**. The carrier $r(z)$ of the element $z=(1+x)^{-1}$ is equal to one. Indeed,

$$z^2(1-r(z))=U_z(1-r(z))=0;$$

applying the operator to this equality $U_{1+x} = U_z^{-1}$, we obtain $1 - r(z) = \theta$, i.e. $r(z) = 1$. Since in the universal semifield every element with support equal to one is invertible, then in the semifield $\bar{A}_0$ exists z^{-1}. Due to the uniqueness of the inverse element in the Jordan algebra.

$$(1+x) = z^{-1} \in \bar{A}_0 \text{, i.e. } x \in \bar{A}_0 \text{, i.e. } K \subset \bar{A}_0 .$$

For any $x \in \bar{A}_0$ we have

$$x = \frac{1}{2}(1+x)^2 - x^2 - 1 \in K - K - K \subset \bar{A}_0 \text{ those. } \widehat{A}_0 = \bar{A}_0 .$$

Thus, we have proved that every maximal strongly associative subalgebra $\widehat{A}$ is a universal semifield . In particular, axioms 3) and (II) *OJ* are algebras for $\widehat{\mathbf{A}}$.

It remains only to verify the fulfillment of the axiom (I). Let be $\{x_\alpha\}$ an increasing network of elements bounded from above in $\widehat{\mathbf{A}}$. We can assume that $\theta \le x_\alpha \le x$ for all α. There is $a = (1+x)^{-1} \in \mathbf{A}$. By virtue of the positivity of the operator U_a in $\widehat{\mathbf{A}}$ and, therefore, in $\widehat{\mathbf{A}}$, the network is $\{U_a x_a\}$ increasing and bounded from above by the element $U_a x = (1+x)^{-2} x \le 1$. Therefore, $\{U_a x_a\} \subset \mathbf{A}$ and therefore in **A** exists $b = \sup U_a x_a$. Then, obviously, the element $x_0 U_a^{-1} b = U_{1+x} b$ is the least upper bound for $\{x_a\}$.

Let us show that $x_a \to x_0$ in the topology t . Since $U_a x_a \uparrow b$ in *JBW*- algebra **A** and for monotone networks in *JBW* - algebras, the concepts of ordinal, *- weak and strong convergence coincide, then $U_a x_a \to b$ strongly, i.e. $\rho((U_a x_a - b)^2) \to 0$ for any normal state ρ . In particular, $\tau((U_a x_a - b)^2) \to 0$ for any $\varepsilon, \delta_1 > 0$ there exists a_0 such that $\tau((U_a x_a - b)^2) \le \varepsilon^2 \delta_1$ for $a \ge a_0$. From here, as in the proof of Theorem 1.8.3, it follows that $(U_a x_a - b) \in N(\varepsilon, \delta)$, with respect to the subadditive measure, i.e., $U_a x_a \to b$ in the topology *t*. Since multiplication in the Jordan algebra $\widehat{\mathbf{A}}$ is continuous in the topology *t*, then

$$x_a = U_{1+x} U_a x_a \xrightarrow{t} U_{1+x} b = x_0$$

If now $y \in \widehat{\mathbf{A}}$ and $y \leftrightarrow x_a$ for any *a* , then, due to the continuity of multiplication in $\widehat{\mathbf{A}}$ and the fact that $x_a \xrightarrow{t} x_0$, it follows that $y \leftrightarrow x_0$, which proves the fulfillment of the $\widehat{\mathbf{A}}$ axioms (I) *OJ* - algebras. The theorem has been proven.

It follows from this theorem that in the case of finite *JBW*-algebras *OJ*- algebras of measurable elements constructed from the trace [1] and from finite subadditive measures coincide.

LITERATURE

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